DeWitt-Virasoro construction

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Abstract

We study a particular approach for analyzing worldsheet conformal invariance for bosonic string propagating in a curved background using hamiltonian formalism. We work in the Schrödinger picture of a single particle description of the problem where the particle moves in an infinite-dimensional space. Background independence is maintained in this approach by adopting DeWitt's (Phys.Rev.85:653-661,1952) coordinate independent formulation of quantum mechanics. This enables us to construct certain background independent notion of Virasoro generators, called DeWitt-Virasoro (DWV) generators, and invariant matrix elements of an arbitrary operator constructed out of them in spin-zero representation. We show that the DWV algebra is given by the Witt algebra with additional anomalous terms that vanish for Ricci-flat backgrounds. The actual quantum Virasoro generators should be obtained by first introducing the vacuum state and then normal ordering the DWV generators with respect to that. We demonstrate the procedure in the simple cases of flat and pp-wave backgrounds. This is a shorter version of arXiv:0912.3987 [hep-th] with many technical derivations omitted.

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Contents

1 Introduction and summary

Equations of motion (EOM) for backgrounds in string theory are derived from the condition of worldsheet conformal invariance[1]. This has mostly been studied by computing the beta functions of the nonlinear sigma model using background field method [1]. A BRST hamiltonian approach was also discussed to some extent in the literature [7, 8, 9, 10, 11, 12, 13] (see also [14]) where one attempts to verify the constraint algebra at the quantum level[2].

In this work we explore a new approach for studying the same problem in hamiltonian framework. In this approach one attempts to find a suitable string generalization of DeWitt's coordinate independent description of particle quantum mechanics as discussed in [20, 21][3]. We attempt to develop the framework in analogy with ordinary particle quantum mechanics where, given a classical hamiltonian, the quantum mechanical problem can be solved in two steps, namely (1) find the quantum hamiltonian using which the Schrödinger equation needs to be written down, (2) find the vacuum (and other excited states) by solving the Schrödinger equation. In case the second step can not be carried out exactly, one develops a perturbative method for solving the problem approximately. The first step is enough to provide the complete quantum definition for a single particle

[1]Some of the original references are [1, 2, 3, 4, 5, 6].

[2]Recently a similar method has been used [15, 16, 17, 18] to study exact conformal invariance of the worldsheet theory in type IIB R-R plane-wave background [19].

[3]DeWitt's formulation was applied to string theory earlier in [22].

system. However, for a quantum field theoretic system, as the case at hand, it serves the purpose only in a limited sense because of the divergences arising from the infinite number of degrees of freedom. Such divergences need to be cured by introducing the vacuum state. In this work we will address the question analogous to the first step only leaving the general study of the vacuum for future work. However, we will discuss the latter in the specific cases of flat and pp-wave backgrounds. Finding out how to follow through the second step in general is an interesting question that needs further investigation. One may also wonder at this point how even the first step itself can be carried out in presence of divergences. As we will see, manifest general covariance will enable us to manipulate various expressions containing such divergences.

We will now describe our construction in detail. This is done by first developing a single particle description of the worldsheet theory. At the classical level such a description is obtained by re-writing the theory in terms of suitable Fourier modes such that it describes a particle moving in an infinite-dimensional curved space (subject to certain potential)[4]. This is the kind of non-linear system that was studied in [20, 21]. The general coordinate transformation (GCT) in the infinite-dimensional spacetime (induced by the same in the physical spacetime where the string is moving) is interpreted as a point canonical transformation in the single particle classical mechanical problem. The latter can then be identified as a subgroup of all unitary transformations in the corresponding quantum theory. DeWitt's analysis shows how the quantum theory can be written down in a manifestly covariant manner in position representation.

There are a few generalizations involved in our work from DeWitt's original work. The analysis of [20] considered a non-relativistic particle so that the general covariance was sought only for the spatial slice. In our case we adopt the infinite-dimensional language for the matter part of the worldsheet theory in conformal gauge. The resulting particle-like theory looks like a worldline theory with full covariance in spacetime. A more important difference is having infinite number of Virasoro generators instead of only the hamiltonian as in DeWitt's case. Because of the presence of infinite number of dimensions the theory possesses certain *shift* properties which look unusual from the particle point of view. These properties dictate the behavior of the theory under certain shift of the spacetime dimensions, i.e. the string modes. Since Virasoro generators relate different string modes,

[4]This way the infinite number of degrees of freedom of the string is given an interpretation of number of spacetime dimensions.

such shift properties are inherently related to the existence of these generators.

The above generalization enables us to construct a background independent quantum[5] version of the Virasoro generators, hereafter called DeWitt-Virasoro (DWV) generators, and coordinate invariant matrix elements involving them between two arbitrary scalar states. As expected, all such expressions contain divergences. But such divergences are hidden in the form of infinite-dimensional traces and therefore our expressions can be manipulated in the formal sense. We then go ahead and compute the algebra of DWV generators in spin-zero representation. Interestingly, we find that the result is given by the Witt algebra with additional operator anomaly terms that vanish for Ricci-flat backgrounds. A few comments are in order,

- The central charge term is expected to appear once the vacuum is introduced and the DWV generators are properly normal ordered. We demonstrate this for flat and pp-wave backgrounds.

- The Ricci-anomaly term is a quantum contribution at the leading order in $\hbar = \alpha'$. Such a term arises because of our covariant treatment. This is an interesting result as Ricci-flatness is also found to be the condition for one-loop conformal invariance in the lagrangian framework [1]. However, since a perturbative expansion has not yet been formulated in our work, *a priori* it is not clear how to relate the two results in a precise way.

- As mentioned in footnote 2, exact conformal invariance of the worldsheet theory in type IIB R-R plane-wave background was studied using the hamiltonian method in [15, 16, 17, 18]. However, general covariance was not made manifest in these works. It was pointed out in [17] that this non-covariant computation leads to certain Virasoro anomaly terms that suffer from operator ordering ambiguity.

  Here we show that considering the pp-wave as a special case of the present covariant framework improves the understanding of how the above ambiguity may be fixed.

The rest of the paper is organized as follows. The infinite-dimensional language is explained in sec.2. Construction of the DWV generators has been discussed in sec.3. We

[5]We will use the word *quantum* in the limited sense of step (1) above except for the discussion in section 5 where vacuum and normal ordering are discussed for flat and pp-wave backgrounds.

summarize the results for the DWV algebra in sec.4. We discuss the flat and the pp-wave backgrounds as special cases of the present construction in sec.5.

This article is a shorter version of the work in [23]. Many of the technical derivations, which are omitted here, can be found in that work.

2 Mapping to infinite dimensions

We consider a bosonic closed string propagating in a D dimensional curved background, hereafter called the physical spacetime, with metric $G_{\mu\nu}$. We work in the conformal gauge of the worldsheet theory so that the ghosts are given by the standard (b, c) systems. For the purpose of the present work we will be concerned only with the matter part of the theory. The relevant classical lagrangian is given by,

$$L = \frac{1}{2}\oint \frac{d\sigma}{2\pi}\, G_{\mu\nu}(X(\sigma)) \left[\dot{X}^\mu(\sigma)\dot{X}^\nu(\sigma) - \partial X^\mu(\sigma)\partial X^\nu(\sigma)\right] , \tag{2.1}$$

where $\oint \equiv \int_0^{2\pi}$, $\mu = 0, 1, \cdots, D-1$. A dot and a ∂ denote derivatives with respect to worldsheet time-coordinate τ and space-coordinate σ respectively. We recast this lagrangian in a form that describes a single particle moving in an infinite-dimensional curved spacetime subject to certain potential,

$$L(x, \dot{x}) = \frac{1}{2} g_{ij}(x) \left[\dot{x}^i \dot{x}^j - a^i(x) a^j(x)\right] , \tag{2.2}$$

where x^i are the general coordinates of the infinite-dimensional spacetime. The index i is given by an ordered pair of indices,

$$i = \{\mu, m\} , \tag{2.3}$$

where $m \in Z$ is the string-mode-number such that[6],

$$\begin{aligned} x^i &= \oint \frac{d\sigma}{2\pi}\, X^\mu(\sigma) e^{-im\sigma} , \\ g_{ij}(x) &= \oint \frac{d\sigma}{2\pi}\, G_{\mu\nu}(X(\sigma)) e^{i(m+n)\sigma} , \\ a^i(x) &= \oint \frac{d\sigma}{2\pi}\, \partial X^\mu(\sigma) e^{-im\sigma} . \end{aligned} \tag{2.4}$$

We will mainly work using the infinite-dimensional language. Below we discuss certain properties of this language that will be relevant for our study.

[6]Throughout the paper we will make the following type of index identifications: $i = \{\mu, m\}$, $j = \{\nu, n\}$, $k = \{\kappa, q\}$.

1. We have claimed that the worldsheet theory (2.1) has an interpretation to be generally covariant in the infinite-dimensional sense. To see this explicitly let us consider a GCT in the physical spacetime: $X^\mu \to X'^\mu(X)$ with Jacobian matrix $\Lambda^\mu{}_\nu(X) = \frac{\partial X'^\mu}{\partial X^\nu}$ and its inverse $\Lambda_\mu{}^\nu(X) = \frac{\partial X^\nu}{\partial X'^\mu}$. This induces a GCT in the infinite-dimensional spacetime: $x^i \to x'^i$ such that the Jacobian matrix $\lambda^i{}_j(x) = \frac{\partial x'^i}{\partial x^j}$ and its inverse $\lambda_i{}^j(x) = \frac{\partial x^j}{\partial x'^i}$ are given by,

$$\lambda^i{}_j(x) = \oint \frac{d\sigma}{2\pi}\, \Lambda^\mu{}_\nu(X(\sigma))e^{i(n-m)\sigma} \ , \quad \lambda_i{}^j(x) = \oint \frac{d\sigma}{2\pi}\, \Lambda_\mu{}^\nu(X(\sigma))e^{i(m-n)\sigma} \ . \quad (2.5)$$

One can then show (see [23]) that $g_{ij}(x)$ and $a^i(x)$ transform as tensors,

$$g'_{ij}(x') = \lambda_i{}^k(x)\lambda_j{}^{k'}(x)g_{kk'}(x) \ , \quad a'^i(x') = \lambda^i{}_j(x)a^j(x) \ . \quad (2.6)$$

2. Using the map in (2.4) one can relate any field in the infinite-dimensional spacetime constructed out of the metric, its inverse, $a^i(x)$ and their derivatives to a non-local worldsheet operator. A class of examples, which will prove to be useful for us, is given by a multi-indexed object $u^{i_1j_1\cdots}_{i_2j_2\cdots}(x)$ constructed out of the metric, its inverse, their derivatives and $a^i(x)$ (but not its derivatives) such that $u^{i_1j_1\cdots}_{i_2j_2\cdots}(x)$ can not be factored into pieces which are not contracted with each other. In this case one can construct a local worldsheet operator $U^{\mu_1\nu_1\cdots}_{\mu_2\nu_2\cdots}(X(\sigma))$ simply by performing the following replacements in the expression of $u^{i_1j_1\cdots}_{i_2j_2\cdots}(x)$,

$$g_{ij}(x) \to G_{\mu\nu}(X(\sigma)) \ , \quad g^{ij}(x) \to G^{\mu\nu}(X(\sigma)) \ , \quad \partial_i \to \partial_\mu \ , \quad a^i(x) \to \partial X^\mu(\sigma) \ . (2.7)$$

As was shown in [23], the two objects $u^{i_1j_1\cdots}_{i_2j_2\cdots}(x)$ and $U^{\mu_1\nu_1\cdots}_{\mu_2\nu_2\cdots}(X(\sigma))$ are related to each other by the following general rule,

$$u^{i_1j_1\cdots}_{i_2j_2\cdots}(x) \sim [2\pi\delta(0)]^N \oint \frac{d\sigma}{2\pi}\, U^{\mu_1\nu_1\cdots}_{\mu_2\nu_2\cdots}(X(\sigma))e^{i(m_2+n_2+\cdots)\sigma - i(m_1+n_1+\cdots)\sigma} \ , \quad (2.8)$$

where N is the number of traces in u and the argument of the Dirac delta function $\delta(0)$ appearing on the right hand side is the worldsheet space direction:

$$\delta(0) = \lim_{\sigma\to\sigma'} \delta(\sigma-\sigma') = \lim_{\sigma\to\sigma'} \frac{1}{2\pi}\sum_{n\in Z} e^{in(\sigma-\sigma')} \ . \quad (2.9)$$

The way one gets N factors of $\delta(0)$ on the right hand side is as follows: Each infinite-dimensional trace breaks up into a trace in the physical spacetime which appears in

the expression of U, and a sum over all the string modes which gives rise to a factor of $\sum_{n\in Z} 1 = 2\pi\delta(0)$. We relate the two sides of (2.8) by the symbol $\sim$ to indicate that such a manipulation is understood only at a formal level. The relation (2.8) implies that u enjoys the same tensorial properties in the infinite-dimensional spacetime as U does in the physical spacetime (provided g_{ij} and a^i have the right tensorial property, which is indeed the case as we have already discussed).

3. In the infinite-dimensional language the problem at hand possesses certain shift properties which can be written as:

$$\begin{aligned} u^{i_1+ii_2\cdots}_{j_1j_2\cdots} &= u^{i_1i_2+i\cdots}_{j_1j_2\cdots} = u^{i_1i_2\cdots}_{j_1-ij_2\cdots} = u^{i_1i_2\cdots}_{j_1j_2-i\cdots} = \cdots \ , \\ \partial_{j+l}a^{k+l}(x) &= \partial_j a^k(x) + i(l)\delta^k_j \ , \end{aligned} \tag{2.10}$$

where the factor of i in the second term of the last equation is the imaginary number. Given the spacetime index i as in footnote 6, we have defined $(i) = m$. A shift in the infinite-dimensional index is defined to be $i+j = \{\mu, m+n\}$[7]. It is now obvious that the first relation of (2.10) is a direct consequence of (2.8). The second relation can be obtained from the following one:

$$\partial_j a^k = i(j)\delta^k_j \ . \tag{2.11}$$

The easiest way to get this is to notice that the definitions in (2.4) imply that the infinite dimensional model in (2.2) corresponds to the string worldsheet theory only for the linear profile $a^k(x) = (k)x^k$. Alternatively, one can directly calculate the left hand side of (2.11) using the third equation in (2.4) and,

$$\partial_i = \oint \frac{d\sigma}{2\pi}\, e^{im\sigma}\frac{\delta}{\delta X^\mu(\sigma)} \ , \quad \frac{\delta X^\mu(\sigma)}{\delta X^\nu(\sigma')} = 2\pi\delta^\mu_\nu\delta(\sigma-\sigma') \ . \tag{2.12}$$

The result is given by $in\delta_{n,q}\delta^\kappa_\nu = i(j)\delta^k_j$.

3 DeWitt-Virasoro generators

The goal of this section is to arrive at the background independent version of the quantum Virasoro generators. We will start with the standard expressions for the classical EM

[7]Notice that we choose the physical spacetime index corresponding to $i+j$ by the one associated with the first index (i.e. i) appearing in the shift. We will follow this convention in all our expressions.

tensor and write the classical Virasoro generators in the infinite-dimensional language. Then after quantizing the system we will use DeWitt's argument to define the quantum DWV generators.

The right and left moving components of the classical EM tensor are given by,

$$\begin{aligned} \mathcal{T}(\sigma) &= \frac{1}{4}\left(K(\sigma) - Z(\sigma) + V(\sigma)\right) = \sum_{m\in Z} L_m e^{im\sigma} \ , \\ \tilde{\mathcal{T}}(\sigma) &= \frac{1}{4}\left(K(\sigma) + Z(\sigma) + V(\sigma)\right) = \sum_{m\in Z} \tilde{L}_m e^{-im\sigma} \ , \end{aligned} \tag{3.13}$$

respectively, where,

$$\begin{aligned} K(\sigma) &= G^{\mu\nu}(X(\sigma))P_\mu(\sigma)P_\nu(\sigma) = \sum_{m\in Z} K_m e^{im\sigma} \ , \\ Z(\sigma) &= 2\partial X^\mu(\sigma)P_\mu(\sigma) = \sum_{m\in Z} Z_m e^{im\sigma} \ , \\ V(\sigma) &= G_{\mu\nu}(X(\sigma))\partial X^\mu(\sigma)\partial X^\nu(\sigma) = \sum_{m\in Z} V_m e^{im\sigma} \ . \end{aligned} \tag{3.14}$$

The conjugate momentum is given by: $P_\mu = G_{\mu\nu}(X)\dot{X}^\nu$. It is related to the momentum in the infinite-dimensional language, i.e. $p_i = g_{ij}(x)\dot{x}^j$ according to the same rule in (2.8). The classical Virasoro generators L_m and $\tilde{L}_m$ can now be expressed in terms of the Fourier modes K_m, Z_m and V_m, which can, in turn, be expressed in the infinite-dimensional language. The results are as follows:

$$4L_{(i)} = K_{(i)} - Z_{(i)} + V_{(i)} \ , \quad 4\tilde{L}_{(i)} = K_{(\bar{i})} + Z_{(\bar{i})} + V_{(\bar{i})} \ , \tag{3.15}$$

where we have defined $\bar{i} = \{\mu, -m\}$ and,

$$K_{(i)} = g^{kl+i}(x)p_k p_l \ , \quad Z_{(i)} = 2a^{k+i}(x)p_k \ , \quad V_{(i)} = g_{kl}(x)a^k(x)a^{l+i}(x) \ . \tag{3.16}$$

Notice that a Virasoro generator is a scalar, but has a string mode index $(i) = m$ which, in the infinite-dimensional language, appears to be a shift of the spacetime index, as evident from eqs.(3.16).

Poisson brackets of the generators in (3.15) should satisfy the classical Virasoro algebra. Notice that since GCT is a canonical transformation which preserves the Poisson brackets, it should be possible to write such brackets in a manifestly covariant manner. This has been shown in [23].

We now quantize the system:

$$[\hat{x}^i, \hat{p}_j] = i\alpha'\delta^i_j \ . \tag{3.17}$$

We work in the Schrödinger picture so that the operators do not have explicit τ dependence. The idea is to keep the general covariance manifest in the quantum theory. GCT of any operator independent of momenta does not suffer from any ordering ambiguity and therefore straightforward to compute. Transformation of the momentum operator, which preserves the canonical commutation relations, is taken to be [20, 21]:

$$\hat{p}_i \to \hat{p}'_i = \frac{1}{2}(\lambda_i{}^j(\hat{x})\hat{p}_j + \hat{p}_j \lambda_i{}^j(\hat{x})) \ . \tag{3.18}$$

This defines GCT of an arbitrary operator constructed out of the phase-space variables.

Let us now introduce the position eigenbasis $|x\rangle$. The orthonormality and completeness conditions read:

$$\langle x|x'\rangle = \delta(x,x') = g^{-1/2}(x)\delta(x-x') \ , \qquad \int dw \ |x\rangle\langle x| = 1 \ , \tag{3.19}$$

where $\delta(x-x')$ is the Dirac delta function, $dw = dx g^{1/2}(x)$ and $g(x) = |\det g_{ij}(x)|$. The position space representation of the momentum operator is given by [20],

$$\langle x|\hat{p}_i|x'\rangle = -i\alpha' \left[\partial_i + \frac{1}{2}\gamma_i(x)\right] \delta(x,x') \ , \tag{3.20}$$

where γ_i are the contracted Christoffel symbols[8],

$$\gamma_j = \gamma^i_{ji} \ , \quad \gamma^i_{jk} = \frac{1}{2} g^{il} \left(\partial_j g_{lk} + \partial_k g_{lj} - \partial_l g_{jk}\right) \ . \tag{3.22}$$

Using $\gamma^*_i(x) = \gamma_{\bar{i}}(x)$ and $\partial_{x^i}\delta(x,x') = -(\partial_{x'^i} + \gamma_i(x))\delta(x,x')$ (see [20]) it is straightforward to check that the position space representation in (3.20) is compatible with the following hermiticity properties:

$$(\hat{x}^i)^\dagger = \hat{x}^{\bar{i}} \ , \quad (\hat{p}_i)^\dagger = \hat{p}_{\bar{i}} \ . \tag{3.23}$$

To construct the DWV generators we first define following [25]:

$$\hat{\pi}_j = \hat{p}_j + \frac{i\alpha'}{2}\gamma_j(\hat{x}) \ , \quad \hat{\pi}^\star_j = \hat{p}_j - \frac{i\alpha'}{2}\gamma_j(\hat{x}) \ . \tag{3.24}$$

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[8]Notice that γ_i has a trace and therefore is divergent:

$$\gamma_i(x) = 2\pi\delta(0) \oint \frac{d\sigma}{2\pi} \ \Gamma_\mu(X(\sigma)) e^{im\sigma} \ , \tag{3.21}$$

where the expression for $\Gamma_\mu(X)$ can be read out from (3.22) using the replacement (2.7).

Using (3.18) it is easy to show that these objects are transformed by left and right multiplications respectively under GCT:

$$\hat{\pi}_i \to \hat{\pi}'_i = \lambda_i{}^j(\hat{x})\hat{\pi}_j \ , \quad \hat{\pi}^\star_i \to \hat{\pi}'^\star_i = \hat{\pi}^\star_j \lambda_i{}^j(\hat{x}) \ , \tag{3.25}$$

The quantum definition of the operators in eqs.(3.16) are given by,

$$\begin{aligned} \hat{K}_{(i)} &= \hat{\pi}^\star_k g^{kl+i}(\hat{x})\hat{\pi}_l \ , \\ \hat{Z}_{(i)} &= \hat{Z}^L_{(i)} + \hat{Z}^R_{(i)} \ , \\ \hat{V}_{(i)} &= g_{kl}(\hat{x})a^k(\hat{x})a^{l+i}(\hat{x}) \end{aligned} \tag{3.26}$$

where,

$$\hat{Z}^L_{(i)} = \hat{\pi}^\star_k a^{k+i}(\hat{x}) \ , \quad \hat{Z}^R_{(i)} = a^{k+i}(\hat{x})\hat{\pi}_k . \tag{3.27}$$

Given the transformation properties in (3.25), it is clear that all the operators in (3.26) and (3.27) are invariant under GCT and have the right classical limit. The left-right symmetric combination for $\hat{Z}_{(i)}$ considered in (3.26) gives the right hermiticity property for the DWV generators. These operators give covariant results in the following sense. Consider the matrix element of an arbitrary operator constructed out of these generators between any two scalar states. The result written in position space representation is manifestly covariant.

4 DeWitt-Virasoro algebra

Here we will present the algebra satisfied by the DWV generators $\hat{L}_{(i)}$ and $\hat{\bar{L}}_{(i)}$ defined through eqs.(3.15, 3.26, 3.27). As mentioned earlier, for a generic background the current framework allows us to calculate this algebra only in the spin-zero representation. The details of the computation can be found in [23]. The final results, written in the infinite-dimensional language, are as follows,

$$\langle\chi| \left\{ \begin{array}{l} [\hat{L}_{(i)}, \hat{L}_{(j)}] = (i-j)\alpha' \hat{L}_{(i+j)} + \hat{A}^R_{(i)(j)} \ , \\ [\hat{\bar{L}}_{(i)}, \hat{\bar{L}}_{(j)}] = (i-j)\alpha' \hat{\bar{L}}_{(i+j)} + \hat{A}^L_{(i)(j)} \ , \\ [\hat{L}_{(i)}, \hat{\bar{L}}_{(j)}] = \hat{A}_{(i)(j)} \end{array} \right\} |\psi\rangle \ , \tag{4.28}$$

where $|\chi\rangle$ and $|\psi\rangle$ are two arbitrary scalar states (τ-dependent). The above result is the Witt algebra (without the central charge terms) with additional operator anomaly terms

given by,

$$\begin{aligned}
\hat{A}^R_{(i)(j)} &= 0 \ , \\
\hat{A}^L_{(i)(j)} &= 0 \ , \\
\hat{A}_{(i)(j)} &= \frac{\alpha'^2}{8}\left(\hat{\pi}^{\star k+i} r_{kl}(\hat{x}) a^{l+\bar{j}}(\hat{x}) - a^{k+i}(\hat{x}) r_{kl}(\hat{x}) \hat{\pi}^{l+\bar{j}}\right) \ ,
\end{aligned} \tag{4.29}$$

where $r_{ij}(x)$ is the Ricci tensor in the infinite-dimensional spacetime which, according to the general map (2.8), is related to the same in physical spacetime, namely $R_{\mu\nu}(X)$ in the following way,

$$r_{ij}(x) \sim 2\pi\delta(0) \oint \frac{d\sigma}{2\pi} \ R_{\mu\nu}(X(\sigma)) e^{i(m+n)\sigma} \ . \tag{4.30}$$

Notice that there are no central charge terms in (4.28). As mentioned in section 1, this is due to the fact that the vacuum, which is a special spin-zero state, has not been introduced and the DWV generators have not been normal ordered. In general such a procedure is not understood to us. However, we will demonstrate this in the specific examples of flat and pp-wave backgrounds in the next section.

5 Flat and pp-wave backgrounds as special cases

We will discuss the flat and the pp-wave cases separately below in subsections (5.1) and (5.2) respectively. Before that we make some comments based on general grounds. For both the backgrounds the EM tensor does not involve any non-trivial ordering between fields and conjugate momenta in the chosen coordinate system. Another way of seeing this is that in both the cases we have,

$$\gamma_k(x) = 0 \ , \quad g(x) = 1 \ . \tag{5.31}$$

This indicates that any operator calculation done using the canonical commutators without worrying about manifest general covariance should be reliable. This justifies recognizing the computation done in [15, 16, 17] as a special case of the present analysis as will be done below. It also turns out as a consequence of (5.31) that the algebra in (4.28) can be considered as operator equations.

5.1 Flat background

In this case the DWV generators are given, in the usual worldsheet language, as follows,

$$\hat{L}^{(0)}_m = \frac{1}{2}\eta_{\mu\nu}\sum_{n\in Z}\hat{\Pi}^\mu_{m-n}\hat{\Pi}^\nu_n\ , \quad \hat{\tilde{L}}^{(0)}_m = \frac{1}{2}\eta_{\mu\nu}\sum_{n\in Z}\hat{\tilde{\Pi}}^\mu_{m-n}\hat{\tilde{\Pi}}^\nu_n\ , \tag{5.32}$$

where the superscript (0) refers to flat background and,

$$\begin{aligned} \hat{\Pi}^\mu_m &= \frac{1}{\sqrt{2}}\oint\frac{d\sigma}{2\pi}(\eta^{\mu\nu}\hat{P}_\nu(\sigma) - \partial\hat{X}^\mu(\sigma))e^{-im\sigma}\ , \\ \hat{\tilde{\Pi}}^\mu_m &= \frac{1}{\sqrt{2}}\oint\frac{d\sigma}{2\pi}(\eta^{\mu\nu}\hat{P}_\nu(\sigma) + \partial\hat{X}^\mu(\sigma))e^{im\sigma}\ , \end{aligned} \tag{5.33}$$

are the usual creation-annihilation operators,

$$[\hat{\Pi}^\mu_m, \hat{\Pi}^\nu_n] = \eta^{\mu\nu}\alpha'\delta_{m+n,0}\ , \quad [\hat{\tilde{\Pi}}^\mu_m, \hat{\tilde{\Pi}}^\nu_n] = \eta^{\mu\nu}\alpha'\delta_{m+n,0}\ . \tag{5.34}$$

The DWV generators differ from the actual quantum Virasoro generators $\hat{\mathcal{L}}^{(0)}_m$ and $\hat{\tilde{\mathcal{L}}}^{(0)}_m$ by additive c-numbers,

$$\hat{L}^{(0)}_m = \hat{\mathcal{L}}^{(0)}_m + c_m\ , \quad \hat{\tilde{L}}^{(0)}_m = \hat{\tilde{\mathcal{L}}}^{(0)}_m + \tilde{c}_m\ , \tag{5.35}$$

where,

$$\hat{\mathcal{L}}^{(0)}_m = \frac{1}{2}\eta_{\mu\nu}\sum_{n\in Z} :\hat{\Pi}^\mu_{m-n}\hat{\Pi}^\nu_n:\ , \quad \hat{\tilde{\mathcal{L}}}^{(0)}_m = \frac{1}{2}\eta_{\mu\nu}\sum_{n\in Z} :\hat{\tilde{\Pi}}^\mu_{m-n}\hat{\tilde{\Pi}}^\nu_n:\ , \tag{5.36}$$

and $c_m = \tilde{c}_m$ is non-zero only when $m = 0$, in which case it is a divergent constant. The normal ordering :: used in the above equations, which matter only for the Virasoro zero modes, is defined as the oscillator normal ordering with respect to the vacuum $|0\rangle$ defined by,

$$\left.\begin{array}{c} \hat{\Pi}^\mu_m \\ \hat{\tilde{\Pi}}^\mu_m \end{array}\right\}|0\rangle = 0\ , \quad \forall m \geq 0\ . \tag{5.37}$$

It is easy to check using (5.34) that the generators in (5.32) satisfy,

$$\begin{aligned} [\hat{L}^{(0)}_m, \hat{L}^{(0)}_n] &= (m-n)\alpha'\hat{L}^{(0)}_{m+n}\ , \\ [\hat{\tilde{L}}^{(0)}_m, \hat{\tilde{L}}^{(0)}_n] &= (m-n)\alpha'\hat{\tilde{L}}^{(0)}_{m+n}\ , \\ [\hat{L}^{(0)}_m, \hat{\tilde{L}}^{(0)}_n] &= 0\ , \end{aligned} \tag{5.38}$$

which is simply the DWV algebra in (4.28) for flat background. However, the same method of computation applied to $[\hat{\mathcal{L}}^{(0)}_m, \hat{\mathcal{L}}^{(0)}_{-m}]$ gives a result that has operator ordering ambiguity. The result is $2m\alpha'\hat{\mathcal{L}}^{(0)}_0$ up to an additive c-number contribution that can not be calculated using this method because of the ambiguity. As indicated in [26], this c-number contribution, which turns out to be the central charge term, can be found unambiguously by calculating, for example, $\langle 0|\hat{\mathcal{L}}^{(0)}_m \hat{\mathcal{L}}^{(0)}_{-m}|0\rangle$ with $m > 0$,

$$
\begin{aligned}
[\hat{\mathcal{L}}^{(0)}_m, \hat{\mathcal{L}}^{(0)}_n] &= (m-n)\alpha'\hat{\mathcal{L}}^{(0)}_{m+n} + \frac{D}{12}(m^3-m)\alpha'^2\delta_{m+n,0} \ , \\
[\hat{\tilde{\mathcal{L}}}^{(0)}_m, \hat{\tilde{\mathcal{L}}}^{(0)}_n] &= (m-n)\alpha'\hat{\tilde{\mathcal{L}}}^{(0)}_{m+n} + \frac{D}{12}(m^3-m)\alpha'^2\delta_{m+n,0} \ , \\
[\hat{\mathcal{L}}^{(0)}_m, \hat{\tilde{\mathcal{L}}}^{(0)}_n] &= 0 \ .
\end{aligned} \tag{5.39}
$$

5.2 Explaining conformal invariance for pp-wave

In [17] we considered a restricted ansatz for an off-shell pp-wave in type IIB string theory which includes the R-R plane-wave background. The R-R flux part of the background involves the Green-Schwarz fermions on the worldsheet. We will ignore this fermionic part and consider only the bosonic part of the computation which corresponds to switching on a metric-background where the non-trivial components of the metric (in physical spacetime) are given by,

$$
G_{+-} = 1 \ , \quad G_{++} = K(\vec{X}) \ , \quad G_{IJ} = \delta_{IJ} \ , \tag{5.40}
$$

where the vector sign refers to the transverse part with index I. The only non-trivial component of the Ricci-tensor is,

$$
R_{++} \propto \vec{\partial}^2 K \ . \tag{5.41}
$$

The worldsheet theory is expected to be an exact CFT when R_{++} vanishes [27]. We call the background in (5.40) simply as pp-wave[9].

It was argued in [17] that the operator anomaly terms of the Virasoro algebra suffer from an ordering ambiguity and therefore proving conformal invariance was not completely

[9] The particular solution of $R_{++} = 0$ given by,

$$
K = \sum_I s_I X^I X^I \ , \quad \sum_I s_I = 0 \ , \tag{5.42}
$$

is called plane-wave.

settled. We argued below eqs.(5.31) why it should be possible to consider this computation as a special case of the present background independent formulation. Here we would like to show how doing this explains the conformal invariance in the present case resolving the ambiguous situation in the previous work.

The first step is to relate the DWV generators specialized to the present background with the quantum Virasoro generators defined in [17]. One can check that this relation is precisely the same as that in (5.35),

$$\hat{L}^{pp}_m = \hat{\mathcal{L}}^{pp}_m + c_m \ , \quad \hat{\tilde{L}}^{pp}_m = \hat{\tilde{\mathcal{L}}}^{pp}_m + \tilde{c}_m \ , \tag{5.43}$$

where the superscript *pp* refers to the pp-wave being considered. According to the calculations of the present work the algebra satisfied by $\hat{L}^{pp}_m$ and $\hat{\tilde{L}}^{pp}_m$ is given by (4.28) evaluated for the pp-wave. Following the same procedure as in flat-case, which took us from eqs.(5.38) to eqs.(5.39), one finds the following quantum Virasoro algebra in the present case,

$$\begin{aligned}
[\hat{\mathcal{L}}^{pp}_m, \hat{\mathcal{L}}^{pp}_n] &= (m-n)\alpha' \hat{\mathcal{L}}^{pp}_{m+n} + \frac{D}{12}(m^3-m)\alpha'^2 \delta_{m+n,0} + \hat{A}^R_{mn} \ , \\
[\hat{\tilde{\mathcal{L}}}^{pp}_m, \hat{\tilde{\mathcal{L}}}^{pp}_n] &= (m-n)\alpha' \hat{\tilde{\mathcal{L}}}^{pp}_{m+n} + \frac{D}{12}(m^3-m)\alpha'^2 \delta_{m+n,0} + \hat{A}^L_{mn} \ , \\
[\hat{\mathcal{L}}^{pp}_m, \hat{\tilde{\mathcal{L}}}^{pp}_n] &= \hat{A}_{mn} \ ,
\end{aligned} \tag{5.44}$$

where the operator anomaly terms $\hat{A}^R_{mn} = \hat{A}^R_{(i)(j)}$, $\hat{A}^L_{mn} = \hat{A}^L_{(i)(j)}$ and $\hat{A}_{mn} = \hat{A}_{(i)(j)}$ are given by eqs.(4.29) evaluated for the pp-wave.

We will now compare the result in (5.44) found in this work with the one in [17]. The computation of [17] was done using the local worldsheet language. The result is precisely the local version of (5.44) with the local operator anomaly terms given by,

$$\begin{aligned}
&\hat{A}^R_{there}(\sigma,\sigma') = \hat{A}^L_{there}(\sigma,\sigma') = \hat{A}_{there}(\sigma,\sigma') \\
&\propto \Big[\hat{\mathcal{O}}(\sigma,\sigma')\hat{P}_-(\sigma')\hat{P}_-(\sigma') - \hat{\mathcal{O}}(\sigma',\sigma)\hat{P}_-(\sigma)\hat{P}_-(\sigma)\Big]\, \delta_\epsilon(\sigma-\sigma') \\
&- \Big[\hat{\mathcal{O}}(\sigma,\sigma')\partial\hat{X}^+(\sigma')\partial\hat{X}^+(\sigma') - \hat{\mathcal{O}}(\sigma',\sigma)\partial\hat{X}^+(\sigma)\partial\hat{X}^+(\sigma)\Big]\, \delta_\epsilon(\sigma-\sigma') \ ,
\end{aligned} \tag{5.45}$$

where the above three anomaly terms are defined in eqs.(3.20) of [17] and

$$\hat{\mathcal{O}}(\sigma,\sigma') = \hat{P}_I(\sigma)\partial_I K(\hat{\vec{X}}(\sigma')) + \partial_I K(\hat{\vec{X}}(\sigma'))\hat{P}_I(\sigma) \ . \tag{5.46}$$

Therefore comparing (5.44) with the corresponding result in [17] we are able to relate the operator anomaly terms in (5.44) and in (5.45),

$$\hat{A}^R_{(i)(j)} = \oint \frac{d\sigma}{2\pi}\frac{d\sigma'}{2\pi}\ \hat{A}^R_{there}(\sigma,\sigma') e^{-im\sigma - in\sigma'} \ ,$$

$$
\begin{aligned}
\hat{A}^L_{(i)(j)} &= \oint \frac{d\sigma}{2\pi}\frac{d\sigma'}{2\pi}\ \hat{A}^L_{there}(\sigma,\sigma')e^{im\sigma+in\sigma'}\ , \\
\hat{A}_{(i)(j)} &= \oint \frac{d\sigma}{2\pi}\frac{d\sigma'}{2\pi}\ \hat{A}_{there}(\sigma,\sigma')e^{-im\sigma+in\sigma'}\ .
\end{aligned}
\tag{5.47}
$$

It was explained in [17] that the expression in (5.45) suffers from an operator ordering ambiguity. The idea here is to resolve this ambiguity by borrowing the results found here. Therefore, using eqs.(5.47) and (4.29) one concludes that both $\hat{A}^R_{there}(\sigma,\sigma')$ and $\hat{A}^L_{there}(\sigma,\sigma')$ should vanish. Moreover, comparing the equations in (4.29), (5.45) and (5.47) one concludes that the Ricci-term obtained in the present analysis was missing earlier. As we will show below, this is an apparent discrepancy which may be resolved by showing that the relevant term vanishes, though the Ricci tensor itself does not, for the pp-wave under consideration because of certain special properties of the background[10].

The non-trivial components of the infinite-dimensional metric corresponding to (5.40) are,

$$
g_{i_+j_-} = \delta_{m+n,0}\ , \quad g_{i_+j_+} = \oint \frac{d\sigma}{2\pi}\ K(\vec{X}(\sigma))e^{i(m+n)\sigma}\ , \quad g_{i_\perp j_\perp} = \delta_{IJ}\delta_{m+n,0}\ , \tag{5.48}
$$

where the infinite-dimensional spacetime index is divided in the following way: $i = (i_+, i_-, i_\perp)$ such that,

$$
i_+ = \{+, m\}\ , \quad i_- = \{-, m\}\ , \quad i_\perp = \{I, m\}\ . \tag{5.49}
$$

The only non-trivial components of the Ricci tensor are $r_{i_+j_+}(\vec{x})$ (the vector sign referring to the transverse indices $i_\perp$) which is related to $R_{++}(\vec{X})$ in (5.41) according to (4.30). Let us now go back to the last equation in (4.28). By going to the position space representation, integrating by parts and using the shift property (2.10) one can show that the Ricci-term is proportional to,

$$
\int dw\ \chi^* \nabla^{k+i}(r_{kl}a^{l+\bar{j}})\psi = \int dx\ \chi^* g^{k_++ik'_-}\partial_{k'_-}(r_{k_+l_+}a^{l_++\bar{j}})\psi\ , \tag{5.50}
$$

where on the right hand side we have evaluated the term for the pp-wave. This vanishes as the quantity inside the round brackets is independent of x^{i_-}.

[10]This argument, however, will rely on the scalar expectation value of the algebra in (4.28), and not operator equation.

6 Conclusions

In this work we have explored a new approach to study two dimensional non-linear sigma model in hamiltonian formalism. By re-writing the problem in terms of the Fourier modes of the string we develop a language which formally describes motion of a particle in an infinite-dimensional curved background. A background independent notion of the Virasoro generators, called DeWitt-Virasoro generators, has been defined for an arbitrary metric-background (not necessarily on-shell) by following DeWitt's coordinate independent description of quantum mechanics. The algebra of such generators in spin-zero representation has been shown to satisfy the Witt algebra with additional operator anomaly terms that vanish for Ricci-flat backgrounds. The same result has been shown to be true in [24] by constructing tensor representations in a certain sense.

The invariant matrix elements computed in this work involve arbitrary spin-zero states. The field theoretic divergences are hidden in our formal expressions as infinite-dimensional traces. However, because of manifest general covariance one is able to carry out the necessary derivations, at least at a formal level. Such divergences are supposed to get cured by first introducing the right vacuum state and then normal ordering the DWV generators. The central charge terms are also expected to show up in the algebra when such a procedure is carried out. Although this has not been understood in general in the present work, it has been demonstrated for the special cases of flat and pp-wave backgrounds. Understanding of the vacuum state and how the DWV anomaly computed here should be related to the standard one-loop beta function result is important for further progress.

References

[1] D. Friedan, Phys. Rev. Lett. **45**, 1057 (1980); D. H. Friedan, Annals Phys. **163**, 318 (1985); L. Alvarez-Gaume, D. Z. Freedman and S. Mukhi, Annals Phys. **134**, 85 (1981).

[2] C. Lovelace, Phys. Lett. B **135**, 75 (1984); P. Candelas, G. T. Horowitz, A. Strominger and E. Witten, Nucl. Phys. B **258**, 46 (1985); S. Jain, R. Shankar and S. R. Wadia, Phys. Rev. D **32**, 2713 (1985).

[3] E. S. Fradkin and A. A. Tseytlin, Phys. Lett. B **160**, 69 (1985); E. S. Fradkin and A. A. Tseytlin, Nucl. Phys. B **261** (1985) 1.

[4] A. Sen, Phys. Rev. D **32**, 2102 (1985); A. Sen, Phys. Rev. Lett. **55**, 1846 (1985).

[5] C. G. . Callan, E. J. Martinec, M. J. Perry and D. Friedan, Nucl. Phys. B **262**, 593 (1985); C. G. . Callan, I. R. Klebanov and M. J. Perry, Nucl. Phys. B **278**, 78 (1986);

[6] B. E. Fridling and A. Jevicki, Phys. Lett. B **174**, 75 (1986).

[7] J. Maharana and G. Veneziano, Nucl. Phys. B **283**, 126 (1987).

[8] R. Akhoury and Y. Okada, Phys. Lett. B **183**, 65 (1987); R. Akhoury and Y. Okada, Phys. Rev. D **35**, 1917 (1987); R. Akhoury and Y. Okada, Nucl. Phys. B **318**, 176 (1989).

[9] A. K. Das and S. Roy, Z. Phys. C **36**, 317 (1987); A. K. Das, J. Maharana and S. Roy, Phys. Rev. D **40**, 4037 (1989); A. K. Das, J. Maharana and S. Roy, Nucl. Phys. B **331**, 573 (1990).

[10] S. Fubini, J. Maharana, M. Roncadelli and G. Veneziano, Nucl. Phys. B **316**, 36 (1989).

[11] S. Alam, Phys. Rev. D **40**, 4047 (1989).

[12] M. Diakonou, K. Farakos, G. Koutsoumbas and E. Papantonopoulos, Phys. Lett. B **247**, 273 (1990); M. Diakonou, K. Farakos, G. Koutsoumbas and E. Papantonopoulos,

[13] J. Wess, Nucl. Phys. Proc. Suppl. **15**, 35 (1990).

[14] S. Jain, G. Mandal and S. R. Wadia, Phys. Rev. D **35**, 778 (1987); S. Jain, G. Mandal and S. R. Wadia, Phys. Rev. D **35**, 3116 (1987).

[15] Y. Kazama and N. Yokoi, JHEP **0803**, 057 (2008) [arXiv:0801.1561 [hep-th]].

[16] P. Mukhopadhyay, JHEP **0811**, 034 (2008) [arXiv:0807.0923 [hep-th]].

[17] P. Mukhopadhyay, JHEP **0905**, 059 (2009) [arXiv:0902.3750 [hep-th]].

[18] P. Mukhopadhyay, Phys. Rev. D **80**, 126007 (2009) [arXiv:0907.2155 [hep-th]].

[19] M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, JHEP **0201**, 047 (2002) [arXiv:hep-th/0110242].

[20] B. S. DeWitt, Phys. Rev. **85**, 653 (1952).

[21] B. S. DeWitt, Rev. Mod. Phys. **29**, 377 (1957).

[22] A. E. Lawrence and E. J. Martinec, Class. Quant. Grav. **13**, 63 (1996) [arXiv:hep-th/9509149].

[23] P. Mukhopadhyay, arXiv:0912.3987 [hep-th].

[24] P. Mukhopadhyay, arXiv:1004.2396 [hep-th].

[25] M. Omote and H. Sato, Prog. Theor. Phys. **47**, 1367 (1972).

[26] J. Polchinski, "String theory. Vol. 1: An introduction to the bosonic string," *Cambridge, UK: Univ. Pr. (1998) 402 p*

[27] D. Amati and C. Klimcik, Phys. Lett. B **219** (1989) 443; G. T. Horowitz and A. R. Steif, Phys. Rev. Lett. **64**, 260 (1990).